A Variable Step Size of the LMS Algorithm for System Identification

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***Abstract*— The Least Mean Square (LMS) algorithm and adaptive filter techniques have been used in system identification. However, their effectiveness may degrade due to the learning gain in the LMS algorithm is a constant. In this paper, a variable step-size learning parameter (VSSP) is proposed to improve the performance of the unknown system identification. The proposed algorithm is implemented without any offline learning phase, while faster convergence can be achieved. Moreover, the computational complexities of different methods are compared. Comparative simulation results demonstrate the validity of the proposed method.**

Keywords—LMS algorithm, step-size learning parameters, system identification

I. INTRODUCTION

The LMS algorithm was introduced by Widrow [1], and has been applied in many different fields such as active noise control [2, 3], fingerprint identification [4], ZigBee Device Identification [5], etc due to their simplicity in the design and implementation of realistic systems. However, the convergence rate of conventional LMS algorithms may be slow because the adopted constant learning gains (or step size) may not be able to handle the wider operation regimes. To improve the convergence speed, several modifications have been introduced in [6-9]. The adjustment of the step-size parameter in [6] is implemented by minimizing the square of the control error. The simplicity of this algorithm allows us to implement it in practical systems.

Filters can be classified into two types: fixed filter and adaptive filter. Fixed filter is used in applications where the signal and path are known, while adaptive filters are useful when the signal or channel dynamics are unpredictable and change over time. Adaptive filters are widely applied in systems such as recognition, inverse system modeling, equalization, denoising, network audio and echo cancellation, etc [10, 11].

Based on the above considerations, this paper designs a controller that combines an LMS algorithm and an adaptive filter. And then improves system performance by using the variable learning gain μ in [6]. The convergence of the proposed methods is proved. Finally, extensive comparative simulations are carried out to validate the suggested algorithms and to show the improved control response.

II. SYSTEM IDENTIFICATION MODEL

The block diagram of the system identification model is presented in Fig. 1. Where $x(n)$ is the recovery desired signal, $h(n)$ is transmission interference, $P(z)$ is the unknown system, $y(n)$ is the output signal of the adaptive FIR filter and $e(n)$ is the error of the system.

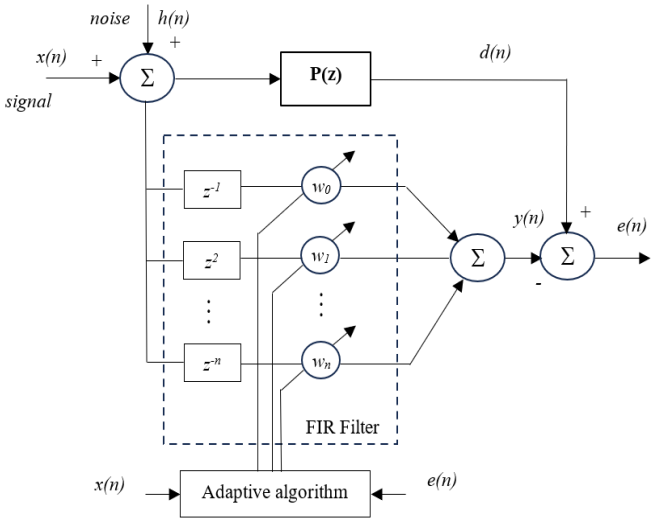


Fig. 1. Structure of the system identification model

A. LMS Algorithm

From Figure 1, the dynamics of the identification system can be described by the following equations. The error of the system is given by

$$e(n) = d(n) - y(n) \quad \text{..... (1)}$$

with

$$y(n) = \sum_{k=0}^{N-1} w_k(n)x(n-k) = w^T(n)X(n) \quad \text{..... (2)}$$

$$w(n) = \left[w_0(n)\ w_1(n), \cdots, w_{N-1}(n)\ \right]^T \quad \text{..... (3)}$$

$$X(n) = \left[x(n), x(n), \cdots, x(n-N-1) \right] \quad \text{..... (4)}$$

Where N is the tap coefficient of the adaptive filter, $X(n)$ is the input of the adaptive controller and $w(n)$ are the parameters of the control $W(z)$, which will be online updated to minimize the aggregate noise $e(n)$.

The LMS algorithm is used to update the weights $w(n)$ of the adaptive control $W(z)$. We define a mean square cost function as $\xi(n) = E\left[e^2(n) \right]$, then the weights $w(n)$ will be

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tuned to minimize the instantaneous error $\xi(n) = e^2(n)$ according to the following gradient-based adaptive law

$$w(n+1) = w(n) - \frac{\mu}{2}\nabla\xi(n) \quad \text{.....(5)}$$

where $\mu > 0$ is the learning gain (step size) of the proposed adaptive algorithm, ∇ is the differential operator as

$$\nabla\xi = \left[\frac{\partial\xi}{\partial w(0)}\ \frac{\partial\xi}{\partial w(1)}, \cdots, \frac{\partial\xi}{\partial w(N-1)}\right]^T \quad \text{.....(6)}$$

The k-th component of $\nabla e^2(n)$ can be given by

$$\frac{\partial}{\partial w(n)}e^2(n) = 2e(n)\frac{\partial e(n)}{\partial w(n)} \quad \text{.....(7)}$$

Because $d(n)$ is independent of $w(n)$ as shown in (1), we substitute $e(n) = d(n) - y(n)$ into (7) and then have the following equation

$$\frac{\partial}{\partial w(n)}e^2(n) = -2e(n)\frac{\partial y(n)}{\partial w(n)} = -2e(n)x(n) \quad \text{....(8)}$$

By substituting (8) into (5), we get the adaptive law for updating the coefficients $w(n)$ as

$$w(n+1) = w(n) + \mu e(n)x(n) \quad \text{.....(9)}$$

Stability analysis

The convergence performance of adaptive law (9) depends on the step size μ. In general, a large μ may improve the convergence speed, while too large μ may trigger instability of the overall control system. In the following, we will first derive the condition to guarantee the control system stability with adaptive law (9). To address the stability of the system with adaptive law (9), we have:

Theorem 1:

For the system shown in Figure 1 with adaptive control (2) and adaptive law (9), if the learning gain fulfills $0 < \mu < 2/\|x(n)\|^2$, then the system is stable.

Proof:

We define a Lyapunov function as follows

$$V_1(n) = \frac{1}{2}e^2(n) = \frac{1}{2}[d(n) - y(n)]^2 \quad \text{.....(10)}$$

Then the difference in the Lyapunov function (10) can be derived as

$$\begin{aligned}\Delta V_1(n) &= V_1(n+1) - V_1(n) = \frac{1}{2}\left[e^2(n+1) - e^2(n)\right] \\ &= \frac{1}{2}[e(n+1) - e(n)][e(n+1) + e(n)] \quad \text{.....(11)} \\ &= \frac{1}{2}\Delta e(n)[2e(n) + \Delta e(n)]\end{aligned}$$

where the error $\Delta e(n)$ for updating the coefficients can be further analyzed as

$$\begin{aligned}\Delta e(n) &= e(n+1) - e(n) \\ &= \left[\frac{\partial e}{\partial w}\right]^T \Delta w(n) = \left[\frac{\partial e}{\partial y}\frac{\partial y(n)}{\partial w(n)}\right]^T \Delta w(n) \quad \text{.....(12)} \\ &= [-x(n)][\mu e(n)x(n)] \\ &= -\mu e(n)\|x(n)\|^2\end{aligned}$$

From (11) and (12), we get

$$\begin{aligned}\Delta V_1(n) &= -\frac{1}{2}\mu e(n)\|x(n)\|^2\left[2e(n) - \mu e(n)\|x(n)\|^2\right] \quad \text{.....(13)} \\ &= -\frac{1}{2}\mu e^2(n)\|x(n)\|^2\left[2 - \mu\|x(n)\|^2\right]\end{aligned}$$

From (10), it is shown that $V(n) = \frac{1}{2}e^2(n) \geq 0$ holds and $V(n) = 0$ if and only if $e(n) = 0$. Then according to the Lyapunov theorem, $V(n)$ will decrease to zero when $\Delta V(n) < 0$ is true. This means that the condition

$$2 - \mu\|x(n)\|^2 > 0 \quad \text{.....(14)}$$

should be fulfilled to guarantee the system's stability.

Consequently, the stability condition of the proposed ANC system can be given as

$$0 < \mu < \frac{2}{\|x(n)\|^2} \quad \text{.....(15)}$$

According to (15), it can be concluded that the adaptive learning gain cannot be set arbitrarily large to retain the system stability. In the following, an online tuning method will be used to update this parameter.

B. VSS-LMS algorithm

The constant learning gain (or step size) in the LMS algorithm (9) may lead to slow convergence speed although stability can be guaranteed by fulfilling condition (15). To accomplish this problem. In this paper, we propose an alternative variable step-size LMS algorithm (VSS-LMS) to modify the classical LMS algorithm used in identification systems.

For this purpose, we propose the modified VSS-LMS algorithm of (9) is given as follows:

$$w_i(n+1) = w_i(n) + 2\mu'(n)e(n)x(n-i) \quad \text{.....(16)}$$

where μ is the variable step-size parameter given as

$$\mu(n) = \begin{cases}\mu_{\max} & \text{if} \quad \mu'(n) > \mu_{\max} \\ \mu_{\min} & \text{if} \quad \mu'(n) < \mu_{\min} \\ \mu'(n) & \text{otherwise}\end{cases} \quad \text{.....(17)}$$

where $\mu_{\max}$ and $\mu_{\min}$ are the maximum and minimum allowable bounds of learning gain, and $\mu'(n)$ is the variable step-size parameter depending on the error $e(n)$, which will be online updated by the following formulation as

$$\mu'(n+1) = \alpha\mu(n) + \gamma e^2(n) \quad \text{.....(18)}$$

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where the constant α is a forgetting factor fulfilling $0 < \alpha < 1$, and the constant $\gamma > 0$ is the learning gain for the adaptation of μ'.

Stability analysis

The performance of the proposed algorithm with adaptive law depends on the learning gain μ' given in (18). Therefore, to ensure the reliability of the proposed algorithm, we identified the following convergence conditions. Developed the formula (18), we obtain

$$\begin{aligned}
\mu'(1) &= \alpha\mu'(0) + \gamma e^2(0) \\
\mu'(2) &= \alpha\mu'(1) + \gamma e^2(1) \\
&= \alpha^2\mu'(0) + \alpha\gamma e^2(0) + \gamma e^2(1) \qquad \text{.....(19)} \\
&\vdots \\
\mu'(k+1) &= \alpha^{k+1}\mu'(0) + \alpha^k\gamma e^2(0) + \\
&+ \alpha^{k-1}\gamma e^2(1) + \cdots \alpha\gamma e^2(k-1) + \gamma e^2(k) \\
&\vdots
\end{aligned}$$

One may verify the property of geometric series

$$\lim_{n\to\infty} \sum_{k=0}^{n-1} \alpha^k = \frac{1}{1-\alpha} \qquad \text{.....(19)}$$

and

$$\lim_{n\to\infty} \alpha^n = 0 \text{ hold for } 0 < \alpha < 1, \qquad \text{.....(20)}$$

then because $|e(n)| \le \varepsilon, n = 0,1,...\infty$ and the initial condition $\mu'(0) \ge 0$ is finite, it can be obtained from **Error! Reference source not found.** that

$$\begin{aligned}
\mu'(\infty) &= \lim_{n\to\infty} \mu'(n+1) \le \lim_{n\to\infty} \alpha^{n+1}\mu'(0) + \lim_{n\to\infty} \sum_{n=0}^{\infty} \alpha^n \gamma\varepsilon \\
&= \frac{\gamma\varepsilon}{1-\alpha} \qquad \text{.....(21)}
\end{aligned}$$

From (21), we can conclude that the variable step size μ' is bounded and will converge to its ideal value as follows

$$\mu^* = \frac{\gamma\varepsilon}{1-\alpha} \qquad \text{.....(22)}$$

From (15), (16) and (22), we easy to see that the convergence conditions of the proposed algorithm is

$$\gamma \le \frac{1-\alpha}{\varepsilon x^2_{\max}(n)} \qquad \text{.....(23)}$$

III. COMPARISONS

To show the efficiency of the proposed methods with the VSS-LMS algorithm, we analyze the computational complexity of the two approaches mentioned in Section II.

(1) Calculate the control output as $y(n) = W^T(n)x(n)$, which requires N multiplications and $N-1$ additions.

(2) Update the filter weight coefficients by $w(n+1) = w(n) + \mu e(n)x(n)$, which requires $N+1$ multiplications and N additions.

(3) Calculate the variable step size by using $\mu'(n+1) = \alpha\mu'(n) + \gamma e^2(n)$, where 2 multiplications, 1 exponentiation, and 1 addition are requested.

Therefore, the computational complexity of the above two control methods is summarized in Table 1.

TABLE I. COMPARISONS OF COMPUTATIONAL COSTS

Controllers	Multiplications	Additions	Exponentiation
LMS	$2N+1$	$2N-1$	
VSS - LMS	$2N+3$	$2N$	1

IV. SIMULATIONS

In this section, to show the effectiveness of the proposed system, we perform experiments for 2 cases: linear system and nonlinear system. In the following simulations, the sampling frequency of the signal is set to be 8 kHz, the order of the adaptive FIR filter is set to 64, the learning gain μ is selected 0.001, and the parameters of the proposed controller based on the VSS-LMS algorithm as $\alpha = 0.98$ and $\gamma = 0.1$. The transmission interference $h(n)$ is selected white noise with the mean = 0 and variance = 0.001.

A. Case 1

In this case, the linear system identification is selected as follows

$$d(n) = 0.5x(n-2) + 0.5x(n-3) \qquad \text{.....(24)}$$

We compare the performance of two different approaches. Fig. 2 shows the input and output signals of the LMS control system. Where the dot–red color presents the input signal and the solid–blue color shows the recovery signal. Figure 3 indicates the displacement between the input signal and the output signal of the VSS-LMS controller.

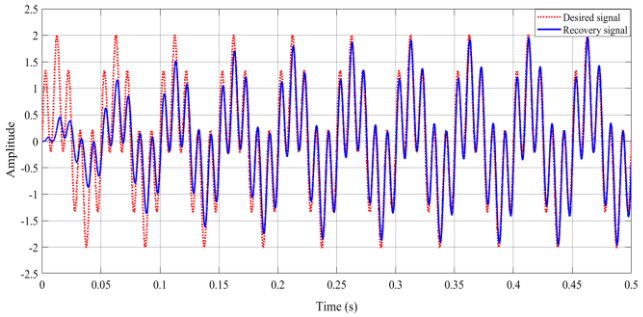


Fig. 2. The displacement of the LMS controller

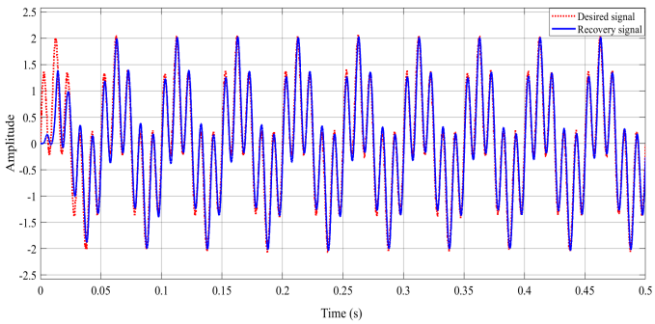
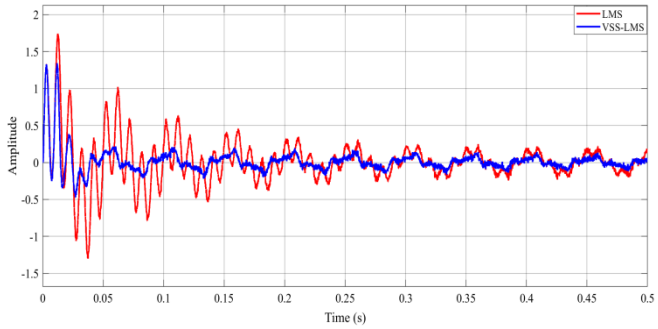


Fig. 3. The displacement of the VSS-LMS controller

Fig. 4 depicts the error signal of systems, where the red line presents the error of the system using the LMS controller and the blue line shows the system error with the VSS-LMS

controller. Here, we can find that the proposed VSS-LMS method achieves better efficiency than the LMS method.



Fig/ 4. Simulation results of system error

Fig. 5 presents the MSE error of systems, where the red line and blue line show the error of the system using the LMS controller and the error of the system with the VSS-LMS controller, respectively. From this figure, we also see the reliability of the proposed VSS-LMS method.

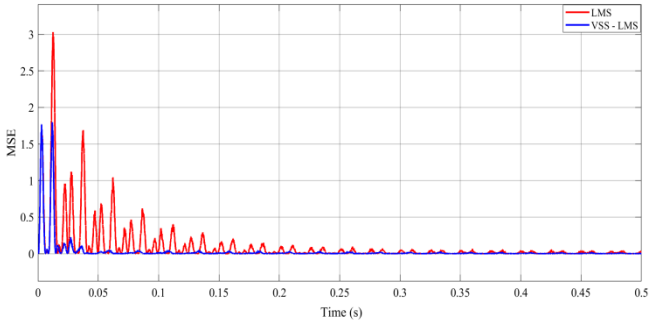


Fig. 5. Simulation results of MSE error

B. *Case 2*

In this case, we select the nonlinear system identification as follows

$$d(n) = 1.5x^2(n-1) + 0.9x(n-5) \quad \text{.....(25)}$$

Simulation results in this case are shown in Fig. 6 ~ Fig. 9. Fig. 6 shows the desired input signal and the output recovery signal of the LMS control method. Fig. 7 presents the simulation results of the VSS-LMS proposed method. Fig. 8 and Fig. 9 show the error and the MSE error of the LMS controller and VSS-LMS controller, respectively. From Fig.7 to Fig. 9, we can confirm the reliability of the proposed VSS-LMS method.

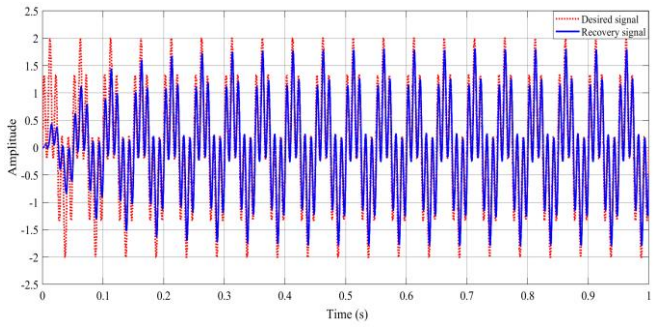


Fig. 6. The displacement of LMS controller for nonlinear system

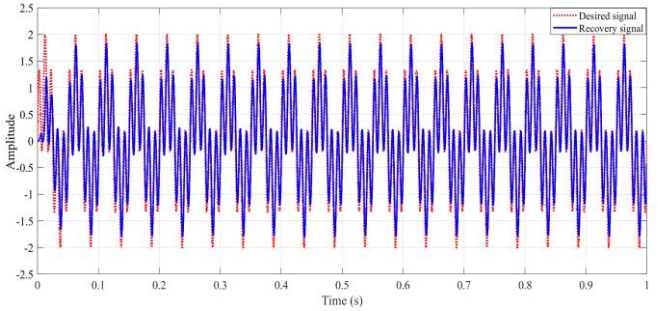


Fig. 7. The displacement of VSS-LMS controller for nonlinear system

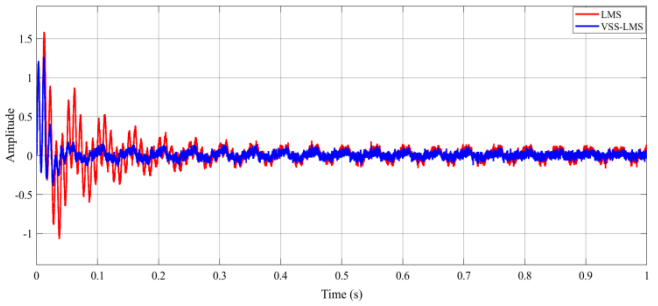


Fig. 8. Simulation results of system error for nonlinear system

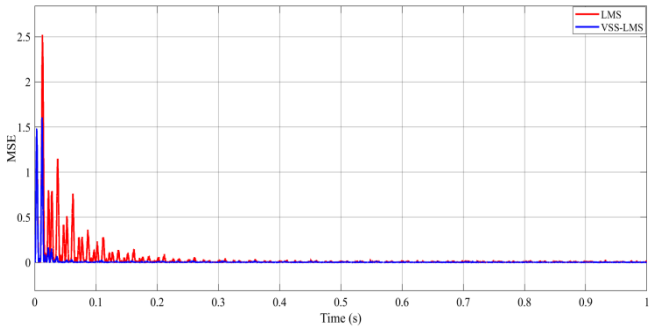


Fig. 9. Simulation results of MSE error for nonlinear system

V. CONCLUSION

In this paper, we propose the VSS-LMS controller for identification systems, where a variable step – size is employed to improve the performance of the system identifications. Compared with the LMS method, the proposed method has superior control performance. Moreover, we also provide the stability and convergence conditions of the proposed method. Finally, the results of different simulation cases also show that the proposed method can obtain an improved control response than the traditional approach.

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